Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
International Journal of Solids and Structures 41 (2004) 7351–7360
INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES
www.elsevier.com/locate/ijsolstr

Explicit approximations of the indentation modulus of elastically orthotropic solids for conical indenters

Antoine Delafargue, Franz-Josef Ulm *

Department of Civil and Environmental Engineering, Massachusetts Institute of Technology, 77 Massachusetts Avenue, Cambridge, MA 02139-4307, USA

Received 30 March 2004; received in revised form 18 June 2004
Available online 14 August 2004

Abstract

The elastic problem of the contact between an axisymmetric indenter and a general anisotropic (21 independent elastic constants) half space has not been solved explicitly in closed form. Implicit methods to determine the indentation modulus originate from the work of Willis [J. Mech. Phys. Solids 14 (1966) 163]; and are now available for conical, parabolic and spherical indenters [Philos. Mag. A 81 (2001) 447; J. Mech. Phys. Solids 51 (2003) 1701]. The particular case of orthotropy has also been investigated [ASME J. Tribol. 115 (1193) 650, 125 (2003) 223]. This paper proposes an explicit solution for the indentation moduli of a transversely isotropic medium and a general orthotropic medium under rigid conical indentation in the three principal material symmetry directions. The half-space Green's functions are interpolated from their exact extreme values, then integrated and finally simplified. The proposed closed form expressions are in very good agreement with the implicit solution schemes of [Philos. Mag. A 81 (2001) 447; J. Mech. Phys. Solids 51 (2003) 1701].

Keywords: Indentation modulus; Orthotropic solid; Surface Green's function; Conical indentation

1. Introduction

Recent developments of nanoindentation techniques make it possible to measure the elastic properties of solids in very small volumes, which is particularly useful for thin films, coatings, composites and heterogeneous materials. For isotropic solids the solution of frictionless conical indentation was found by Love (1939), the interpretation is straightforward and has been used extensively (Oliver and Pharr, 1992). If

* Corresponding author. Tel.: +1 617 253 3544; fax: +1 617 253 6044.
E-mail address: ulm@mit.edu (F.-J. Ulm).

0020-7683/$ - see front matter
doi:10.1016/j.ijsolstr.2004.06.019

simple and explicit relations between stiffness constants and indentation moduli were also available for anisotropic solids, then a few indentation tests performed in various directions would potentially become a standard procedure to characterize the elastic properties of general anisotropic materials. A quick determination of the indentation modulus from known elastic constants would also be useful for many mechanics or materials design aspects.

We restrict our analysis to rigid conical indentation, for which the indentation modulus M is defined from the unloading branch of an indentation test (Doerner and Nix, 1986; Oliver and Pharr, 1992), using the fundamental Hertz contact solution, recast in the form (Vlassak and Nix, 1993):

$$\frac{\mathrm{d}P}{\mathrm{d}h} = \frac{2}{\sqrt{\pi}} M \sqrt{A} \tag{1}$$

where P is the applied load, h is the rigid-body displacement of the indenter relative to the half-space, A is the projected area of contact. For any indentation into a linear elastic half-space, for which the Hertz-type contact possesses classical self-similarity (see Borodich et al., 2003) $\sqrt{A}$ is a linear function of the indentation depth h, and does not depend on the anisotropy of the material. Swadener and Pharr (2001) recognized this property for conical indentation, for which (Fig. 1):

$$\sqrt{A} = \sqrt{\pi} h_c \tan(\alpha) = \frac{2}{\sqrt{\pi}} \tan(\alpha) h \tag{2}$$

where α is the cone half-angle and $h_c = (2/\pi)h$ the contact depth. In turn, for rigid indenters, the indentation modulus M in (1) is a function of only the elastic constants of the indented half-space. M is not a material property but rather a snapshot of the solid stiffness. For instance, in the isotropic case, M reduces to the plane-stress elastic modulus,

$$M = \frac{E}{1 - \nu^2} = \frac{C_{1111}^2 - C_{1122}^2}{C_{1111}} \tag{3}$$

where E is the Young's modulus, ν the Poisson's ratio; C_{1111} and C_{1122} are the forth order stiffness tensor coefficients of the half-space. It is readily understood that the link between the indentation modulus and the elastic constants of a general anisotropic material is far more complicated than in the isotropic case. Closed form solutions, such as (3) are only available for some particular cases, such as the Elliot–Hanson solution (Elliot, 1949; Hanson, 1992) for conical indentation of a transversely isotropic half-space in the axis of symmetry. Much of the recent contributions to the analysis of the Hertzian contact for anisotropic solids can be traced back to the work of Willis (1966), who reduced the problem to the evaluation of contour integrals for parabolic indenters. Vlassak and Nix (1994) simplified the solution using the surface Green's functions determined by Barnett and Lothe (1975) and provided implicit solution schemes for other indenter shapes.

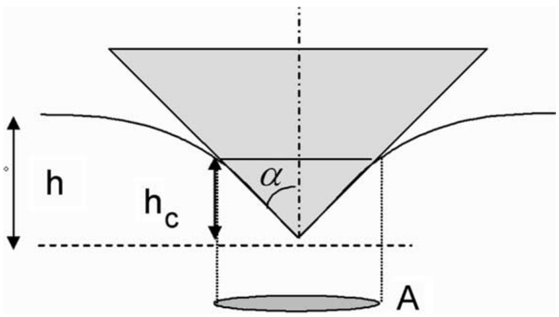

Fig. 1. Indentation and contact depths.

To our knowledge, the most refined solution schemes for general anisotropic materials are those proposed by Swadener and Pharr (2001) for conical and parabolic indenters and by Vlassak et al. (2003) for conical and spherical indenters . Both solution schemes involve computational demanding operations even in their approximated versions. Ovaert (1993) and more recently Shi et al. (2003) investigated the particular cases of transverse isotropy and orthotropy and proposed an implicit resolution methods for ellipsoidal indenters. It is the focus of this paper to develop easy implementable explicit expressions for orthotropic materials indented in the axes of symmetry by conical indenters.

We first give the explicit expression for the indentation modulus of transversely isotropic materials indented in the axis of symmetry, based on the Elliot–Hanson solution. This solution also motivates the extension to indentation normal to the axis of symmetry, based on a sinusoidal approximation of the surface Green's function. Instead of keeping the first two terms of the Fourier transform as suggested by Vlassak et al. (2003), we interpolate the Green's function from some extreme values that are known explicitly. This method does not require the entire expression of the function and thus leads to explicit expressions of the indentation modulus. We finally extend the approach to orthotropic materials, and compare the results for various materials with the results obtained by Swadener and Pharr (2001) and Vlassak et al. (2003).

2. Indentation modulus of a transversely isotropic solid

Let direction $x3$ be normal to the planes of isotropy; directions $x1$ and $x2$ be parallel to the planes of isotropy so that the resulting coordinate system S is a right-hand cartesian one, with the first indented point as origin O.

2.1. Indentation in the axis of symmetry

For transversely isotropic materials, the problem of elastic conical indentation in direction $x3$, i.e. when the plane of isotropy is parallel to the half space surface, has been solved analytically (Elliot, 1949; Hanson, 1992). The problem is axisymmetric, the projected area of contact in the plane $(x1, x2)$ is circular (Fig. 2). The Elliot–Hanson solution reads:

$$P = \frac{2}{\pi^2 H} h^2 \tan(\alpha) \tag{4}$$

H is a constant that depends on the material stiffness constants:

$$H = \frac{1}{2\pi}\sqrt{\frac{C_{11}}{C_{31}^2 - C_{13}^2}\left(\frac{1}{C_{44}} + \frac{2}{C_{31} + C_{13}}\right)} \tag{5}$$

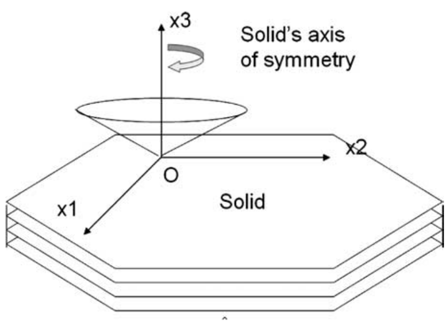

Fig. 2. Indentation in the solid's axis of symmetry.

where we use the reduced notations:

$$\begin{aligned} C_{11} &= C_{1111} \\ C_{13} &= C_{1133} = C_{3311} \\ C_{44} &= C_{2323} = C_{1313} \\ C_{31} &= \sqrt{C_{11}C_{33}} > C_{13}; \quad \text{with } C_{33} = C_{3333} \end{aligned} \tag{6}$$

It is interesting to note that the solution does only depend on four of the five elastic constants of a transversely isotropic material ($C_{11}, C_{33}, C_{44}, C_{13}$). The fifth independent constant $C_{12} = C_{1122} = C_{1111} - 2C_{1212}$ does not appear in the expression of H. We will see that this observation is restricted to the perfect axisymmetric case, for which the factor H turns out to be the constant in the Green's function, i.e. the surface displacement induced by a concentrated unit load:

$$\eta(r) = \frac{H}{r} \tag{7}$$

In general, the Green's function depends on both polar coordinates (r, θ) defined on the indented surface and centered at the load point; but for the perfect material axisymmetric case it does not depend on θ.

Finally, using (4) and (2) in (1) yields the following explicit expression of the indentation modulus M_3 in the axis of symmetry (direction $x3$):

$$M_3 = \frac{1}{\pi H} \tag{8}$$

or equivalently using (5):

$$M_3 = 2\sqrt{\frac{C_{31}^2 - C_{13}^2}{C_{11}}\left(\frac{1}{C_{44}} + \frac{2}{C_{31} + C_{13}}\right)^{-1}} \tag{9}$$

2.2. Indentation normal to the axis of symmetry

When the half-space's surface is orthogonal to the material's planes of isotropy, the area of contact is not circular, and the problem is no longer axisymmetric (Fig. 3).

The load versus displacement relation in such a contact problem can be found in two steps:

1. Finding the displacement field corresponding to a concentrated load; that is the Green's function (Vlassak and Nix, 1994):

$$\eta(\mathbf{y}) = \frac{1}{8\pi^2|\mathbf{y}|}\left[\alpha_k B_{km}^{-1}\left(\frac{\mathbf{y}}{|\mathbf{y}|}\right)\alpha_m\right] = \eta(r, \theta) = \frac{H(\theta)}{r} \tag{10}$$

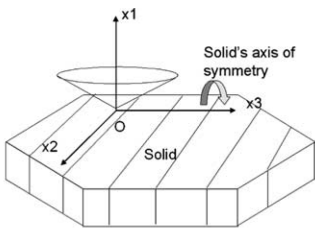

Fig. 3. Indentation orthogonal to the solid's axis of symmetry.

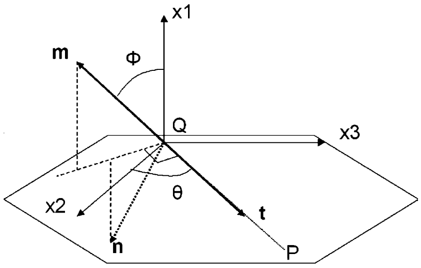

Fig. 4. (m,n,t) coordinates system.

where $\mathbf{y}$ is the position vector of Q relative to the load point P; (r,θ) are polar coordinates of Q ($\theta=0$ along direction $x2$); α_1, α_2, α_3 are the cosines of the direction normal to the indented surface; $\mathbf{B}$ is a second order tensor defined by Barnett and Lothe (1975):

$$B_{js}(\mathbf{t}) = B_{sj}(\mathbf{t}) = \frac{1}{8\cdot\pi^2}\int_0^{2\pi}((\mathbf{mm})_{js} - (\mathbf{mn})_{jk}(\mathbf{nn})^{-1}_{kr}(\mathbf{nm})_{rs})\,\mathrm{d}\phi \tag{11}$$

where $(\mathbf{ab})_{jk} = a_i \mathrm{C}_{ijkl}\mathrm{b}_m$ (with the summation of repeated indices); $\mathbf{t}$ is the normalized form of $\mathbf{y}$; $(\mathbf{m},\mathbf{n},\mathbf{t})$ forms a right hand cartesian system, and ϕ is the angle between vector $\mathbf{m}$ and direction $x1$ (Fig. 4). And finally, $\eta(r,\theta)$ is the surface Green's function, homogenous in r^{-1}.

2. Integrating the Green's function to find the displacement field resulting from an assumed pressure distribution under the indenter (Swadener and Pharr, 2001; Willis, 1966), and verifying that result matches with the boundary conditions. [1]

2.2.1. Green's function approximation

In our particular case of transverse isotropy, the indentation axis $(x1)$ belongs to two planes of symmetry (Fig. 3): $(x1,x3)$ is orthogonal to the planes of isotropy, and $(x1,x2)$ is parallel to them. In this case, an evaluation of (10) for $(\alpha_1,\alpha_2,\alpha_3)=(1,0,0)$ yields the exact values of the Green's function η in the $x2$-direction $(\theta=0)$, and in the $x3$-direction $(\theta=\pi/2)$:

$$H(\theta=0) = \frac{1}{2\pi}\sqrt{\frac{C_{33}}{C_{31}^2 - C_{13}^2}\left(\frac{1}{C_{44}} + \frac{2}{C_{31}+C_{13}}\right)} = H_2 \tag{12}$$

$$H\left(\theta=\frac{\pi}{2}\right) = \frac{1}{\pi}\frac{C_{11}}{C_{11}^2 - C_{12}^2} = H_3 \tag{13}$$

It is interesting to note, from a comparison of (5) and (12), that $H_2 = \sqrt{C_{33}/C_{11}}H$; while H_3 turns out to be the Green's function constant for an isotropic material with stiffness constants C_{11} and C_{12}.

Furthermore, since the Green's function is even and π-periodic by symmetry, H_2 and H_3 are extreme values of $H(\theta)$. Therefore, it seems appropriate to consider a first order approximation $\widetilde{H}(\theta)$ that interpolates the Green's function $\eta(r,\theta)\approx\widetilde{H}(\theta)/r$ so that $\widetilde{H}(\theta=0)=H_2$ and $\widetilde{H}(\theta=\frac{\pi}{2})=H_3$:

[1] An original alternative to the assumption of a pressure distribution was proposed by Vlassak et al. (2003), using Barber's theorem (1975).

$$\widetilde{H}(\theta) = H_0 + H_{c1}\cos(2\theta) \tag{14}$$

where

$$H_0 = \frac{H_2 + H_3}{2} \quad \text{and} \quad H_{c1} = \frac{H_2 - H_3}{2} \tag{15}$$

The proposition of interpolating the Green's function between two extreme values is restricted to the geometrical configuration of indentation in the principal material axes. A more general first order approximation is obtained by performing a Fourier transform of $H(\theta)$, and keeping only the first terms of the sinusoidal decomposition (see Vlassak et al., 2003, Appendix A). This approximation gives a slightly different sine function whose coefficients must be evaluated numerically; whereas H_0, H_{c1} in our solution procedure are found analytically. In return, since the first order approximation from Fourier transform was found to be very accurate for many materials, it should also be the case with our explicit evaluation of the Green's function.

2.2.2. Integration over an assumed pressure distribution

It is generally assumed that the projected contact area is elliptical for conical indentation of general anisotropic materials (Swadener and Pharr, 2001; Vlassak et al., 2003). By symmetry, the axes of the elliptical contact area must coincide with $x2$ and $x3$. If a_2 and a_3 are the ellipse dimensions in the respective directions $x2$ and $x3$, then the ellipse eccentricity is $e = \sqrt{1 - \left(\frac{a_2}{a_3}\right)^2}$ if $a_2 < a_3$, and $e = \sqrt{1 - \left(\frac{a_3}{a_2}\right)^2}$ otherwise. We assume that the pressure field $p(y_2, y_3)$ at point P of coordinates (y_2, y_3) has the form proposed by Swadener and Pharr (2001):

$$p(y_2, y_3) = p_0 \cosh^{-1}\left(\frac{y_2^2}{a_2^2} + \frac{y_3^2}{a_3^2}\right)^{-1/2} \tag{16}$$

The displacement in any point $Q(z_2, z_3)$ situated on the projected contact surface is:

$$u_1(z_2, z_3) = \int\int_S p(y_2, y_3)\eta(z_2 - y_2, z_3 - y_3)\mathrm{d}y_2\,\mathrm{d}y_3 \tag{17}$$

The indentation depth h is equal to the displacement u_1 at the cone tip. It can be expressed in (17) as a function of the load $P = \pi a_2 a_3 p_0$ and identified with (1) and (2), so that M_1 is given by (Swadener and Pharr, 2001; Vlassak et al., 2003):

$$M_1 = \frac{1}{\alpha(e, \Theta)(1 - e)^{1/4}} \tag{18}$$

where:

$$\alpha(e, \Theta) = \int_0^{\pi} \frac{\eta(\theta + \Theta)}{\sqrt{1 - e^2\cos^2(\theta)}}\mathrm{d}\theta \tag{19}$$

Θ is the angle between the major axis of the contact ellipse and the direction $x2$ or $\theta = 0$: by symmetry $\Theta = 0$ if $H_2 < H_3$, $\Theta = \pi/2$ otherwise.

The solution of (18) and (19) requires as input the eccentricity. For our proposed Green's function approximation (14), integration of (17) with (16) yields the displacement field u_1, that is subjected to the condition of axisymmetry imposed by the contact of the rigid indenter. Following faithfully the method described by Swadener and Pharr (2001) and Vlassak et al. (2003, Appendix A), we obtain an explicit expression of the eccentricity:

$$e = \sqrt{\frac{2|H_{c1}|}{H_0 + |H_{c1}|}} = \left\{ \sqrt{1 - \frac{H_2}{H_3}}, \quad \text{if } H_2 < H_3, \sqrt{1 - \frac{H_3}{H_2}}, \text{ else} \right\} \tag{20}$$

or equivalently,

$$\frac{a_2}{a_3} = \sqrt{\frac{H_3}{H_2}} \tag{21}$$

Finally, substituting the eccentricity (20) in (18) and (19), we obtain:

$$M_1 = \frac{1}{2E(e) H_2^{3/4} H_3^{1/4}} \tag{22}$$

where $E(e)$ is the complete elliptic integral of the second kind. It is useful to rewrite (22) in the form:

$$M_1 = \Psi(e) \frac{1}{\pi\sqrt{H_2 H_3}}; \quad \text{where } \Psi(e) = \pi \frac{(1 - e^2)^{1/4}}{2E(e)} \tag{23}$$

We note that $0.99 \leqslant \Psi(e) \leqslant 1$ if $e \leqslant 0.6$, which corresponds to an ellipse axis ratio smaller that 1.25. Hence, using $\Psi(e) \approx 1$ in (23) simplifies the expression of the indentation modulus normal to the axis of symmetry:

$$M_1 \approx \frac{1}{\pi\sqrt{H_2 H_3}} = \sqrt{M_{12} M_{13}} \tag{24}$$

where:

- M_{13} appears as the indentation modulus obtained by indentation in an isotropic solid, for which the elastic properties in direction $x3$ coincide with the elastic properties in direction $x1$ and $x2$:

$$M_{13} = \frac{1}{\pi H_3} = \frac{C_{11}^2 - C_{12}^2}{C_{11}} \tag{25}$$

- M_{12} would be the indentation modulus in direction $x1$, if the elastic properties in direction $x2$ had been set equal to the properties in direction $x3$:

$$M_{12} = \frac{1}{\pi H_2} = \sqrt{\frac{C_{11}}{C_{33}}} M_3 \tag{26}$$

3. Indentation modulus of an orthotropic solid

If we consider the same cartesian coordinates system, such that the orthotropic solid's three planes of symmetry are along $(x1, x2)$, $(x1, x3)$ and $(x2, x3)$, the indentation moduli in the directions $x1$, $x2$ and $x3$ can be approximated using the same method. For example, in the case of indentation in direction $x1$ of surface $(x2, x3)$, there are two perpendicular planes of symmetry, $(x1, x2)$ and $(x1, x3)$.

For an orthotropic solid, we consider the nine independent stiffness constants of the material:

$$\begin{aligned} &C_{11} = C_{1111} \quad \text{or} \quad C_{21} = \sqrt{C_{1111}C_{2222}};\ C_{12} = C_{1122} = C_{2211};\ C_{44} = C_{2323} \\ &C_{33} = C_{3333} \quad \text{or} \quad C_{31} = \sqrt{C_{1111}C_{3333}};\ C_{13} = C_{1133} = C_{3311};\ C_{55} = C_{1313} \\ &C_{22} = C_{2222} \quad \text{or} \quad C_{32} = \sqrt{C_{2222}C_{3333}};\ C_{23} = C_{2233} = C_{3322};\ C_{66} = C_{1212} \end{aligned} \tag{27}$$

Along similar lines of arguments developed before, we obtain:

$$\begin{aligned} M_1 &\approx \sqrt{M_{12}M_{13}} \\ M_2 &\approx \sqrt{M_{21}M_{23}} \\ M_3 &\approx \sqrt{M_{31}M_{32}} \end{aligned} \tag{28}$$

where:

$$\begin{aligned} M_{21} &= 2\sqrt{\frac{C_{21}^2 - C_{12}^2}{C_{11}}\left(\frac{1}{C_{66}} + \frac{2}{C_{21} + C_{12}}\right)^{-1}} \\ M_{31} &= 2\sqrt{\frac{C_{31}^2 - C_{13}^2}{C_{11}}\left(\frac{1}{C_{55}} + \frac{2}{C_{31} + C_{13}}\right)^{-1}} \\ M_{32} &= 2\sqrt{\frac{C_{32}^2 - C_{23}^2}{C_{22}}\left(\frac{1}{C_{44}} + \frac{2}{C_{32} + C_{23}}\right)^{-1}} \end{aligned} \tag{29}$$

and

$$\begin{aligned} M_{12} &= M_{21}\sqrt{\frac{C_{11}}{C_{22}}} \\ M_{13} &= M_{31}\sqrt{\frac{C_{11}}{C_{33}}} \\ M_{23} &= M_{32}\sqrt{\frac{C_{22}}{C_{33}}} \end{aligned} \tag{30}$$

4. Discussion

The presented approximations were tested on several transversely isotropic and orthotropic materials. The results were compared with the 'exact' values obtained by the implicit methods of Swadener and Pharr (2001), and Vlassak et al. (2003). By way of example, we consider an hexagonal crystal of zinc that can be modeled as transversely isotropic. Using the elastic coefficients, $C_{1111} = C_{2222} = 164$, $C_{3333} = 62.93$, $C_{2323} = C_{1313} = 39$, $C_{1122} = 36$, $C_{2233} = C_{1133} = 52$ (GPa), our approximated indentation modulus in the direction parallel to the hexagonal planes is $M_{1x} = 132.2$ GPa, which is very close to the one evaluated using Vlassak et al.'s solution: $M(x1) = 133.4$ GPa. For this particular case, Fig. 5 represents the 'exact' surface Green's function computed from Vlassak et al. (2003), and two first order approximations: the Fourier transform approximation suggested by Vlassak et al. (2003), and our Green's function interpolation approximation (14). Both approximations are close to the target function. By construction, our explicit approximation is exact for $\theta = 0$ and $\theta = \pi/2$. By way of example of an orthotropic material, we determine the indentation modulus for the human tibial cortical bones tested by Swadener et al. (2001), using the con-

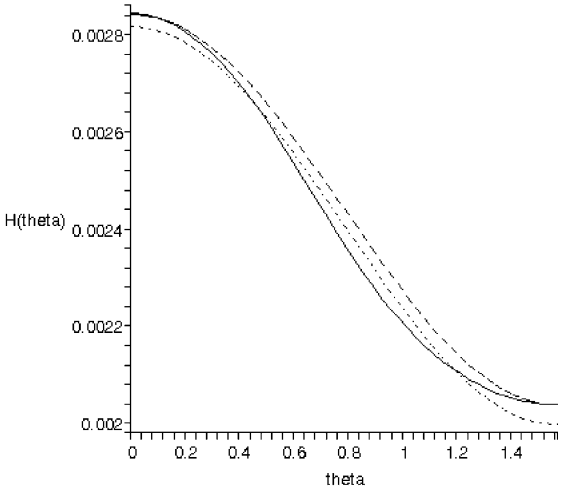

Fig. 5. Green's function of zinc's surface $(x2, x3)$ in 1/GPa (solid line). Our first order approximation (dashed). The equivalent one from a Fourier transform (dotted). Theta varies from 0° to 90°.

stants $C_{1111} = 19.5$, $C_{2222} = 20.1$, $C_{3333} = 30.9$, $C_{2323} = 5.72$, $C_{1313} = 5.17$, $C_{1212} = 4.05$, $C_{2233} = C_{1133} = 12.5$, $C_{12} = 11.4$ (GPa). Our approximation (28) yields:

$$M_1 = 14.0659 \text{ GPa}, M_2 = 14.6090 \text{ GPa}, M_3 = 19.6784 \text{ GPa} \tag{31}$$

which cannot be distinguished from the values given by Swadener and Pharr: $M(x1) = 14$ GPa, $M(x2) = 14.6$ GPa, $M(x3) = 19.7$ GPa.

The high accuracy of the relative simple closed form expressions seems to be due to a combination of two facts: (1) a first order approximation of the Green's function appears to be highly relevant for many materials; and (2) the eccentricities encountered with many materials are rather small. On the other hand, the proposed solutions only apply to the three principal material directions. The full back-analysis from indentation tests of the five (respectively nine) independent elastic constants for transversally isotropic (respectively orthotropic) materials will require three (respectively six) further solutions in inclined directions. However, the solutions in the principal material directions have the premise to display the highest contrast in indentation stiffness. The additional information could also be provided by assumptions about Poisson's ratios and shear moduli for example.

Finally, determining explicit solutions of indentation moduli from known elastic constants can be useful for many mechanics or materials design aspects.

References

Barnett, D.M., Lothe, J., 1975. Line force loadings on anisotropic half-spaces and wedges. J. Appl. Phys. 23, 470–474.

Borodich, F.M., Keer, L.M., Korach, C.S., 2003. Analytical study of fundamental nanoindentation test relations for indenters of non-ideal shapes. Nanotechnology 14, 803–808.

Doerner, M.F., Nix, W.D., 1986. A method for interpreting the data from depth-sensing indentation instruments. J. Mater. Res. 1, 601–609.

Elliot, H.A., 1949. Axial symmetric stress distributions in aelotropic hexagonal crystals: The problem of the plane and related problems. Proc. Camb. Phil. Soc. 45, 621–630.

Hanson, M.T., 1992. The elastic field for conical indentation including sliding friction for transverse isotropy. J. Appl. Mech. 59, S123–S130.

Love, A.E.H., 1939. Boussinesq's problem for a rigid cone. Quarterly Journal of Mathematics 10, 161–175.
Oliver, W.C., Pharr, G.M., 1992. An improved technique for determining hardness and elastic modulus using load and displacement sensing indentation experiments. J. Mater. Res. 7, 1564–1583.
Ovaert, T.C., 1993. On the indentation of a transversely isotropic half-space with application to thin solid lubricant films. ASME J. Tribol. 115, 650–657.
Shi, D., Lin, Y., Ovaert, T.C., 2003. Indentation of an orthotropic half-space by a rigid ellipsoidal indenter. ASME J. Tribol. 125, 223–331.
Swadener, J.G., Pharr, G.M., 2001. Indentation modulus of elastically anisotropic half-spaces by cones and parabolae of revolution. Philos. Mag. A 81, 447–466.
Swadener, J.G., Rho, J.Y., Pharr, G.M., 2001. Effects of anisotropy on elastic moduli measured by nanoindentation in human tibial cortical bone. J. Biomed. Mat. Res. 57, 108–112.
Vlassak, J.J., Nix, W.D., 1993. Indentation modulus of elastically anisotropic half spaces. Phil. Mag. A 67 (5), 1045–1056.
Vlassak, J.J., Nix, W.D., 1994. Measuring the elastic properties of anisotropic materials by means of indentation experiments. J. Mech. Phys. Solids 42 (8), 1223–1245.
Vlassak, J.J., Ciavarella, M., Barber, J.R., Wang, X., 2003. The indentation modulus of elastically anisotropic materials for indenters of arbitrary shape. J. Mech. Phys. Solids 51, 1701–1721.
Willis, J.R., 1966. Herztian contact of anisotropic bodies. J. Mech. Phys. Solids 14, 163–176.